KANT'S THEORY OF KNOWLEDGE AND THE EPISTEMOLOGICAL FOUNDATION OF MATHEMATICS

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Kant's view comes to dominate *West European* philosophy and his *theory of knowledge* plays a crucial role in the *foundation of mathematics*. A clear understanding of his notions of would do much to elucidate his *epistemological approach*. *Kant's theory of knowledge* seems, hitherto, to have been analyzed by *post modern philosophers*, and some *mathematicians,* and it even seems directly to rage their conjectures through incontestably certain in the ultimate concern of its consequences. Perry R.B. retrieves that Kant's contributions to *epistemological foundation of mathematics* consisted in his discovery of *categories* and the *form of thought* as the universal prerequisites of mathematical knowledge. According to Perry[1], in his *Prolegomena to any Future Metaphysics,* Kant exposed a question "*How Is Pure Mathematics Possible?*". While Philip Kitcher[2] in Hersh R. shows that all three *foundationist* gurus Frege, Hilbert, and Brouwer were Kantians; that was a

[1] Perry, R.B., 1912, "*Present Philosophical Tendencies: A Critical Survey of Naturalism Idealism Pragmatism and Realism Together with a Synopsis of the Pilosophy of William James",* New York: Longmans Green and Co. p. 139
[2] Hersh, R., 1997*, "What is Mathematics, Really?"*, London: Jonathan Cape, p.132

consequence of the influences of Kant's philosophy in their early milieus, and the usual tendency of research mathematicians toward an *idealist* vewpoint.

The publication of Kant's great works[3] did not put an end to the *crisis* in the *foundation of philosophy*. On the contrary, they raged about it more furiously than ever. As *two main schools* found in the *philosophy of mathematics*, before and after Kant, the latent elements of them were discovered and brought to the higher level. One school considered as the *sceptic*al promoting of the new analysis, and proceeded to build its dome furnished by its material; the other took advantage of the positions gained by the ultimate champion and developed its lines forward in the direction of *transcendental* claim. Kant[4] lays the *foundations of philosophy*; however, he built *no structure*. He did not put one stone upon another; he declared it to be beyond the power of man to put one stone upon another. Kant[5] attempts to erect a temple on his foundation he repudiated. The existence of an *external world* of substantial entities corresponding to our conceptions could not be demonstrated, but only logically affirmed.

[3] Ibid. p. 129

[4] ….., "*Immanuel Kant, 1724–1804*", Retrieved 2004 <*http://www.alcott.net/alcott/home/champions/Kant.html*>

[5] *Ibid.*

A. Kant's Theory of Knowledge Synthesizes the Foundation of Mathematics

Bolzano B. comments that *Kant's theory of knowledge* seem to promise us with his discovering of a definite and characteristic difference between *two main classes* of all human *a priori knowledge* i.e. *philosophy* and *mathematics.* According to Kant[6] *mathematical knowledge* is capable of adequately presenting all its concepts in a *pure intuition*, i.e. *constructing*. It is also able to demonstrate its *theorems*; while, on the other hand, *philosophical knowledge*, devoid of all intuition, must be content with *purely discursive concepts.* Consequently[7] the *essence* of mathematics may be expressed most appropriately by the *definition* that it is *a science of the construction of concepts.* Bolzano[8] suggests that several *mathematicians* who adhere to the critical philosophy have actually adopted this definition and deserved much credit to *Kant's theory of knowledge* for the *foundation of pure mathematics.*

Kant's theory of knowledge, as it deserved in his *transcendental philosophy*, had a distinct work i.e. the "*Critique of Pure Reason* (1781)", in which he opens a new epoch in *metaphysical thought* where far in the history of philosophy the human mind had not been fairly considered. Thinkers[9] had concerned themselves with the *objects*

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[6] Bolzano, B., 1810, "*Contributions to a Better-Grounded Presentation of Mathematics"* in Ewald, W., 1996, "*From Kant to Hilbert: A Source Book in the Foundations of Mathematics, Volume I*", Oxford: Clarendon Press, p. 175

[7] *Ibid. p. 175*

[8] *Ibid. p.175*

[9] *Immanuel Kant, 1724–1804*", Retrieved 2004 <*http://www.alcott.net/alcott/ home/champions/ Kant. html*>

of knowledge, not with the mind that knows, while *Kant tries to transfer contemplation from the objects that engaged the mind to the mind itself*, and thus start philosophy on a new career. Shabel L. (1998) perceives that epistemologically, Kant primarily regards to determine whether the method for obtaining *apodictic certainty*, that one calls *mathematics* in the latter science or pure reason in its mathematical use, is identical with that by means of which one seeks the same *certainty in philosophy,* and that would have to be called as *dogmatic.*

Prior and after Kant[10], there are questions left for some writers to get their position about the foundation of mathematics e.g. *Whether or not we have the mathematical ideas that are true of necessity and absolutely? Are there mathematical ideas that can fairly be pronounced independent in their origin of experience, and out of the reach of experience by their nature?* One party contends that *all mathematical knowledge* was derived from *experience* viz. there was nothing in the *intellect* that had not previously been in the senses. The opposite party maintains that *a portion of mathematical knowledge* came from the *mind* itself viz. the *intellect* contained powers of its own, and impressed its forms upon the *phenomena of sense*. The extreme doctrine of the *two schools* was represented, on the one side by the *materialists,* on the other by the *mystics.* Between these *two extremes* Kant might be perceived to offer *various degrees of compromise* or raging its foundations spuriously.

The ultimately discussion sums up that, in the sphere of Kant's '*dogmatic*' notions, his *theory of knowledge, i*n turn, can be said to lead to *un-dogmatization* and

[10]Ibid.

de-mythologization of *mathematical foundations* as well as rages the institutionalization of the research of *mathematical foundations* in which it encourages the mutual interactions among them. Smith, N. K. in "*A Commentary to Kant's Critique of Pure Reason*", maintains that some further *analytic explanations* supporting the claims come from Kant's claims that there are *three possible standpoints* in philosophy i.e. *the dogmatic, the sceptical,* and *the critical.* All preceding thinkers[11] come under the first two heads. Kant[12] insists that a *dogmatist* is one who assumes that human reason can comprehend *ultimate reality*, and who proceeds upon this assumption; it expresses itself through three factors viz. *rationalism, realism,* and *transcendence.*

According to Smith, N. K.[13], for Kant, Descartes and Leibniz are *typical dogmatist*s. On the other hand, *rationalists*[14] held that it is possible to determine from *pure a priori principles* the *ultimate nature of the material universe*. They are *realists* in that they assert that by *human thought,* the complete nature of objective reality, can be determined. However, they also adopt the *attitude of transcendence*. Through pure thought[15], *rationalists* go out beyond the sensible and determine the *supersensuous.* Meanwhile[16], *scepticism* may similarly be defined through the three terms, *empiricism*,

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[11] Smith, N. K., 2003, "*A Commentary to Kant's Critique of Pure Reason*", New York: Palgrave Macmillan, p. 13
[12] *Ibid. p. 13*
[13] *Ibid. p. 13*
[14] *Ibid*. p. 13
[15] *Ibid. p.14*
[16] *Ibid. p. 14*

subjectivism, immanence. Further, Smith, N. K[17] clarifies that a *sceptic* can never be a *rationalist*. The *sceptic* must reduce knowledge to *sense-experience*; for this reason also his knowledge is infected by *subjective conditions*. Through *sensation* we cannot hope to determine the nature of the *objectively real.* This attitude is also that of immanence and knowledge is limited to the sphere of *sense-experience.*

Smith, N. K[18] synthesizes that *criticism* has similarly its three constitutive factors, *rationalism, subjectivism, immanence.* Accordingly, it agrees with *dogmatism* in maintaining that only through *a priori principles* can *true knowledge* be obtained. Such *knowledge*[19] is, however, subjective in its origin, and for that reason it is also only of *immanent application*. *Knowledge* is possible only in the sphere of *sense-experience. Dogmatist*[20] claims that *knowledge* arises independently of experience and extends beyond it. *Empiricism*[21] holds that *knowledge* arises out of *sense-experience* and is *valid* only within it; while, *criticism* teaches that *knowledge* arises independently of *particular experience* but is *valid* only for *experience.*

It[22] can be learned from Kant that the *sceptic* is the taskmaster who constrains the *dogmatic reasoner* to develop a sound critique of the *mathematical understanding* and *reason.* The *sceptical procedure*[23] cannot of itself yield any satisfying answer to the questions of *mathematical reason*, but none the less it prepares the way by

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[17] *Ibid. p. 14*
[18] *Ibid. p. 14*
[19] Ibid. p. 14
[20] *Ibid. p.14*
[21] *Ibid. p. 21*
[22] *Ibid. p. 21*
[23] *Ibid. p.21*

awakening its circumspection, and by indicating the radical measures which are adequate to secure it in its legitimate possessions. Kant develops the *method* as, the *first step,* in matters of pure reason, marking its infancy, is *dogmatic*. The *second step* is *sceptic*al to indicate that experience has rendered our *judgment* wiser and more circumspect. The *third step,* is now necessary as it can be taken only by *fully matured judgment*. It is not the *censorship* but *the critiqe of reason* whereby *not its present bounds but its determinate and necessary limits*; *not its ignorance on this or that point but is regard to all possible questions of a certain kind*. *Mathematical reasons* are demonstrated from *principles,* and not merely arrived at by way of *mathematical conjecture*. [24] *Scepticism* is thus *a resting-place* for *mathematical reason,* where it can reflect upon its dogmatic wanderings and make survey of the region in which it finds itself, so that for the future it may be able to choose its part with more certainty.

The role of *Kant's theory of knowledge*, in the sense of *de-mythologization* of *mathematical foundations,* refers to history of the mathematical *myth* from that of Euclid's to that of contemporary philosophy of mathematics. The *myth* of Euclid: *"Euclid's Elements contains truths about the universe which are clear and indubitable"*, however, today advanced student of *geometry* to learn *Euclid's proofs* are *incomplete* and *unintelligible*. Nevertheless, *Euclid's Elements* is still upheld as a *model of rigorous proof.* The *myths* of Russell, Brouwer, and Bourbaki - of *logicism, intuitionism,* and *formalism*-are treated in *The Mathematical Experience.*

[24] *Ibid*. p. 14

Contemporary mathematical foundations[25] consist of general *myths* that: *1. Unity i.e. there is only one mathematics, indivisible now and forever, and it is a single inseparable whole; 2. Universality i.e. the mathematics we know is the only mathematics there can be; 3. Certainty i.e. mathematics has a method, "rigorous proof;" which yields absolutly certain conclusions, given truth of premises; 4. Objectity i.e. mathematical truth is the same for everyone and it doesn't matter who discovers it as well as true whether or not anybody discovers it.*

Kant's theory of knowledge implies to the critical examinations of those *myths*. In fact, being a *myth* doesn't entail its truth or falsity. *Myths* validate and support institutions in which their truth may not be determinable. Those latent mathematical *myths* are almost *universally accepted*, but they are not *sef-evident or self-provin*g. From a different perspective, it is possible to question, doubt, or reject them and some people do reject them. Hersh, R. in *"What is Mathematics, Really?"* indicates that if mathematics were presented in the style in which it is created, few would believe its *universality, unity, certainty*, or *objectivity*. These *myths*[26] support the *institution of mathematics*. While the *purists* sometimes even declare *applied mathematics* is not *mathematics.*

The clarity and strict necessity of *mathematical truth* had long provided the *rationalists* - above all Descartes, Spinoza and Leibniz - with the assurance that, in the world of modern doubt, the human mind had at least *one solid basis* for attaining

[25] Hersh, R., 1997*, "What is Mathematics, Really?"*, London: Jonathan Cape, p. 37
[26] *Ibid. p. 38*

certain knowledge. Kant himself had long been convinced that *mathematics* could accurately describe the *empirical world* because *mathematical principles* necessarily involve a context of *space* and *time*. According to Kant, *space* and *time* lay at the basis of all *sensory experience* i.e. the condition and structure any empirical observation. For Kant, *intuitions* are supposed to be *eternal* and *universal* features of *mind* which constitutes all *human thinking.* While, for *rationalists*, *mathematics was the main example to confirm their view of the world*.

From the three historic schools, the mainstream philosophy of mathematics records only *intuitionism* pays attention to the *construction of mathematics*. *Formalists, Logicists,* and *Platonists* sit at a table in the dining room, discussing their rag out as a self-created, autonomous entity. Smith, N. K. concerns with Kant's conclusion that there is no *dwelling-place* for permanent settlement obtained only through perfect certainly in our *mathematical knowledge*, alike of its objects themselves and of the limits which all our knowledge of object is enclosed. In other word[27], *Kant's theory of knowledge* implies to *un-dogmatization* and *de-mythologization* of *mathematical foundations* as well as to rage the institutionalization of the *research of mathematical foundations.* In term of these perspectives, Kant considers himself as contributing to the further advance of the eighteenth century *Enlightenment* and in the future prospect of *mathematics philosophy*.

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[27] Smith, N. K., 2003, "*A Commentary to Kant's Critique of Pure Reason*", New York: Palgrave Macmillan, p. 21

B. Kant's Theory of Knowledge Contributes to Epistemological Foundation of Mathematics

As Mayer, F., notes that Kant's fundamental questions concerning epistemology covers how are *synthetical judgments a priori* possible and the solution of that problem; and comprehending the possibility of the use of *pure reason* in the *foundation* and *construction* of *all sciences,* including *mathematics*; as well as concerning the solution of this problem depending on the existence or downfall of the science of metaphysics. According to Kant.[28], in a system of absolute, *certain knowledge* can be erected only on a foundation of judgments that are *synthetic*al and acquired independently of all experience. While, Hegel, G.W.F (1873) indicates that Kant's epistemology does not seek to obtain knowledge of the object itself, but sought to clarify how *objective truthfulness* can be obtained, as he named it the "*transcendental method*."

On the other hand, Distante P. recites that epistemologically, Kant attempts a compromise between *empiricism* and *rationalism*. According to Distante P.[29], Kant agrees with the *rationalists* that one can have exact and certain knowledge, but he followed the *empiricists* in holding that such knowledge is more informative about the structure of thought than about the world outside of thought. Further, he

[28] In Mayer, F., 1951, "*A History of Modern Philosophy"*, California: American Book Company, p.295
[29] Distante, P., 2000-2003, "*Epistemology*" Retrieved 2004 *<http://home. earthlink. net/ ~pdistant /index.html>).*

indicates that Kant restricts knowledge to the *domain of experience*, but attributes to the mind a function in incorporating sensations into the structure of experience. This structure could be known *a priori* without resorting to empirical methods. According to Kant[30], mathematics has often been presented as *a paradigm of precision* and *certainty*. It[31], therefore, concerns the way to know the *truth of mathematical propositions,* the *applications of abstract mathematics* in the real world and *the implications of mathematics* for the information revolution, as well as the *contributions of mathematics.* It[32] leads us to examine mathematics as a primary instance of what philosophers have called a priori knowledge.

Steiner R. (2004) thought that in the epistemological sense, Kant has established the *a priori nature of mathematical principles*, however, all that the *Critique of Pure Reason* attempts to show that *mathematics* is a priori sciences. From this, it follows that the *form of all experiences* must be inherent in the subject itself. Therefore[33], the only thing left that is empirically given is the material of sensations. This is built up into a system of experiences, the form of which is inherent in the subject. Kant[34] maintains that *mathematics* is *synthetic a priori*. If *mathematical truths* are known, where can we find the *basis* or *grounding* of their status as knowledge? The only possibility for knowledge of claims, that are not based on definitions, are universal and go beyond experience as if there is *synthetic a priori knowledge.*

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[30] Wilder,R.L., 1952, *"Introduction to the Foundation of Mathematics",* New York, p.192
[31] *Ibid. p.193*
[32] *Ibid. p. 193*
[33] The Rudolf Steiner Archive. Retrieved 2004 <mailto:elibrarian@elib.com>
[34] -----, 2003, *"Kant's Mathematical Epistemology*", Retrieved 2004 <*http://www.wesleyan.edu/phil/ courses /202/s00/ pdfs/ phil202s00_11a.pdf.*>

Hersh R. (1997) assigns that Kant's fundamental presupposition is that contentful knowledge independent of experiences, can be established on the basis of universal human intuition. While Mayer, F. (1951) indicates that based on *apodictic* knowledge forms as the foundation of his philosophy, Kant made it clear that *mathematics*, as universal scientific knowledge, depends on *synthetic judgments a priori*; and claims that *synthetic a priori judgments* are the foundation of mathematics Again, Wilder R.L. (1952) ascertains Kant that *mathematical judgments*, at least the most characteristic ones, were *synthetic*, rather than *analytic*; and argues that *mathematics* is *a pure product of reason*, and moreover is thoroughly *synthetic*al.[35] . However, Posy C. indicates that according to Kant, *mathematics* is about the *empirical world*; it is special in one important way that *necessary properties* of the world are found through mathematical proofs. To prove something is wrong[36], one must show only that the world could be different.

Kant's theory of knowledge [37] states that *mathematics* is basically generalizations from experience, but this can provide only *contingent* of the possible properties of the world. *Mathematics* is about the *empirical world*, but usually methods for deriving knowledge give *contingent knowledge*, not the *necessity* that pure mathematics gives us. Kant [38] wants *necessary knowledge* with *empirical knowledge,* while confirming that the *objects* in the *empirical world* are *appearances*

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[35] Wikipedia The Free Encyclopedia. Retrieved 2004 <*http:/en.wikipedia.org/*>
[36] Posy, C. ,1992, "*Philosophy of Mathematics*". Retreived 2004 <*http://www.cs.washington.edu/ homes/gjb.doc/philmath.htm*>
[37] *Ibid*.
[38] *Ibid.*

or *phenomenon* and therefore we come to know them only from *experiences*. According to Kant[39], in order to know the properties of *mathematical objects* we need to build into our minds two forms of *intuition* and *perception* in such away that every perception we have is conceived by these forms i.e. *space* and *time*. These are, in fact, parts of the mind, and not some-thing the mind picks up from experience; thus, *empirical objects* are *necessarily spatial-temporal* objects.

Still, Posy C. (1992) indicates that Kant insists *mathematics* as the studying of the abstract form of perception or, in other words, *mathematics* is simply the science that studies the *spatial-temporal* properties of objects. Bolzano B. learns Kant**'s** observation that the *principle of sufficient reason* and the majority of *propositions of arithmetic* are *synthetic propositions*; however, who does not feel how artificial it is, has to assert that these propositions are based on *intuitions.* Kant[40] claims that, *in geometry*, there are certain underlying *intuitions*; for in fact, many people may think that the concept of *point* is the *intuition* of a *point* before our eyes. However[41], the picture accompanying our *pure concept of the point* is not connected with it but only through the association of *ideas*; in fact, we have often thought both of them together.

Bolzano B[42], on the other hand, claims that the nature of this *geometrical picture* is different with different people; it is determined by thousands of fortuitous

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[39] *Ibid.*

[40] Bolzano, B., 1810, "*Appendix: On the Kantian Theory of the Construction of Concepts through Intuitions*" in Ewald, W., 1996, "*From Kant to Hilbert: A Source Book in the Foundations of Mathematics, Volume I",* Oxford: Clarendon Press, p.223

[41] *Ibid.p. 223*

[42] *Ibid*. p.223

circumstances. However, Kant[43] adds that if we had always seen just roughly and thickly drawn *lines* or had always represented *a straight line* by chains or sticks, we would have in mind with the *idea of a line* i.e. *the image of* a chain or a stick. Kant[44] said: *"With the word 'triangle' one always has in mind an equilateral triangle, another a right-angled triangle, a third perhaps an obtuse-angled triangle".* According to Kant[45], *mathematical judgments* are *all synthetical;* however he argues that this fact seems hitherto to have altogether escaped the observation of those who have analyzed human reason. It even seems directly opposed to all their conjectures, though incontestably certain, and most important in its consequences.

Kant in *"Prolegomena to Any Future Metaphysics",* claims that the conclusions of mathematicians proceed according to the *law of contradiction*, as is demanded by all *apodictic certainty*. Kant[46] says that it is a great mistake for men persuaded themselves that the *fundamental principles* were known from *the same law*. Further, Kant[47] argues that *the reason* that for a *synthetic*al *proposition* can indeed be comprehended according to *the law of contradiction* but only by presupposing another *synthetical proposition* from which it follows. Further, Kant[48] argues that *all principles of geometry* are *no less analytical*; and that the proposition "*a straight line*

[43] *Ibid.p.223*
[44] *Ibid. p.223*
[45] Kant, I, 1783, "*Prolegomena to Any Future Metaphysic:* REMARK 1 Trans. Paul Carus. Retrieved 2003 <*www. phil-books.com/* >
[46] *Ibid*
[47] *Ibid.*
[48] *Ibid.*

is the shortest path between two points" is a *synthetical proposition* because *the concept of straight* contains *nothing of quantity*, but only *a quality*.

Kant[49] claims that the *attribute of shortness* is therefore altogether *additional*, and cannot be obtained by any analysis of the concept; and its *visualization* must come to aid us; and therefore, it alone makes the synthesis *possible.* Kant[50] then confronts the *previous geometers* assumption which claims that *other mathematical principles* are indeed actually *analytic*al and depend on *the law of contradiction*. Kant strives to show that *identical propositions* such as "*a=a", " the whole is equal to itself",* or *"a + b > a", "the whole is greater than its part", etc,* is a method of concatenation, and not the principles. Kant then claims that although they are recognized as valid from mere concepts, they are only admitted in mathematics, because they can be represented in some visual form. Hersh R. reveals that *Kant's theory of spatial intuition* means *Euclidean geometry* was *inescapable*. But the establishment of *non-Euclidean geometry* gives us choices. While Körner[51] says Kant didn't deny the abstract conceivability of *non-Euclidean geometries*; he thought they could never be realized in real *time* and *space*

It may need to hold Faller's notions[52] that *Kant's theory of knowledge* most significantly contributes to the *foundation of mathematics* by its recognition that

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[49] *Ibid*

[50] Kant, I, 1783, "*Prolegomena to Any Future Metaphysic:* REMARK 1 Trans. Paul Carus. Retrieved 2003 <*www. phil-books.com/* >

[51] Hersh, R., 1997*, "What is Mathematics, Really?"*, London: Jonathan Cape, pp.132

[52] Faller, M., 2003, "*Kant's Mathematical Mistake*", Retrieved 2004 <*http://polar.alas kapa i fic.edu/mfaller/ KntMth.PDF.>*

mathematical knowledge holds that *synthetic a priori judgments* were possible. Kant[53] recognizes that *mathematical knowledge* seems to bridge the *a priori analytic* and *a posteriori synthetic*. According to Kant, *mathematical thinking is a priori in the universality, necessity of its results and synthetic in the expansively promise of its inquiry*. Particularly, Wilder R.L.(1952) highlights that *Kant's view* enables us to obtain a more accurate picture of the *role of intuition* in mathematics. However, at least as developed above, it is not really satisfying, because it takes more or less as a fact our ability to place our perceptions in *a mathematically defined structure* and to see truths about this structure by using perceptible objects to symbolize it.

According to Wilder R.L.[54], Kant's restriction his discussion to parts of cognition could ground such knowledge to epistemological elaboration of the basis of *synthetic a priori* knowledge of mathematics. Kant[55] contributes the solution by claiming that *geometric propositions* are *universally valid* and *must be true of all possible objects of experience*. It is not enough that *all triangles* we have seen have a given property, but *all possible triangles* we might see must have it as well. According to Kant[56], epistemologically *there are two ways to approach the foundation of mathematics:* first, *perceiving that there is something about the world that makes it so;* second, *perceiving that there are something about our experiences that makes it so*. The *first* alone[57] can not produce knowledge because an *objective mind-independent*

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[53] *Ibid.*
[54] *Ibid*.
[55] *Ibid.*
[56] *Ibid.*
[57] *Ibid.*

fact might be *universally true*, but we could never verify its universality by experience. *So the only source of the foundation for mathematics lies in the second alternative i.e. there is something about our experiencing that makes it so.*

Meanwhile, Wilder R.L[58] issues that, in the epistemology of arithmetic, e.g. in *Kant's verification of 7+5=12*, one must consider it as an *instance* i.e. *this time in the form of a set of five objects, and add each one in succession to a given set of seven.* Although the *five objects* are *arbitrary*, they will be represented by the symbols which are present and which exhibit the same structure; and contemporary, we find this structure involved in the *formal proofs of 7+5=12* either within a set theory or directly from axioms for elementary number theory. The proofs in the set theory depend on *existential axioms* of these theories.

Meanwhile, Shabel L. believes that Kant explores an *epistemological explanation* whether *pure geometry* ultimately provides a structural description of certain features of empirical objects. According to Shabel L.[59], Kant requires his first articulation that *space* is *a pure form of sensible intuition* and argues that, in order to explain the *pure geometry* without *paradox*, one must take the concept of *space* to be *subjective*, such that it has its source in our cognitive constitution. Kant[60] perceives that *epistemological foundation of geometry* is only possible under the *presupposition* of a given way of explaining our pure intuition of *space* as the form of our *outer sense.*

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[58] Wilder, R. L. , 1952, *"Introduction to the Foundation of Mathematics",* New York, p. 197

[59] Shabel, L., 1998, "*Kant's "Argument from Geometry*", Journal of the History of Philosophy, The Ohio State University, p.19

[60] *Ibid. p.20*

In term of the theory of the epistemology of spatial objects, Kant[61] denies that we use geometric reasoning to access our *pure intuition of space*, in favor of affirming that we use our pure intuition of *space* to attain geometric knowledge. Kant[62] claims that *pure spatial intuition* provides an epistemic starting point for the practice of geometry. Therefore the *pure spatial intuition* constitutes an *epistemological foundation* for the mathematical disciplines.

Ultimately, for Kant[63] and his contemporaries, the *epistemological foundations of mathematics* consists amount of a view to which our *a priori* mental representation of *space-temporal intuition* provides us with the original cognitive object for our mathematical investigations, which ultimately produce *a mathematical theory of the empirical world*. However[64], Kant's account of *mathematical cognition* serves still remains unresolved issues. Shabel L.[65] concludes that the great attraction of *Kant's theory of knowledge* comes from the fact that other views seem unable to do any better. Frege, for example, carries the epistemological analysis less than Kant in spite of his enormously more refined logical technique.

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[61] *Ibid.p.34*
[62] *Ibid. p.34*
[63] *Ibid.p.34*
[64] *Ibid*. p.34
[65] In Wilder, R. L. , 1952, *"Introduction to the Foundation of Mathematics",* New York, p.205

C. Kant's Theory of Sensible Intuition Contributes to Constructive and Structural Mathematics

For Kant[66], to set up the *foundation of mathematics* we need to start from the very initially step analysis of *pure intuition*. Kant means by a *"pure intuition"* as an intuition purified from particulars of experience and conceptual interpretation. i.e., we start with experience and abstract away from concepts and from particular sensations. The impressions made by *outward thing* which is regarded as *pre-established forms* of sensibility i.e. *time* and *space*. *Time*[67] is no empirical conception which can be deduced from experience and *a necessary representation* which lies at the foundation of all intuitions. It is given *a priori* and, in it alone, is any reality of phenomena possible; it disappears, but it cannot be annihilated. *Space* is an intuition, met with in us *a priori*, *antecedent* to any perception of objects, a pure, not an empirical intuition. These two *forms of sensibility*, inherent and invariable to all experiences, are subject and prime facts of consciousness in the *foundation of mathematics.*

Wilder R.L. issues that, for Kant, *sensible intuition* was *necessary* in the *foundation of mathematics.* According to Kant [68], the *a priori character of mathematical judgments* is *synthetic*, rather than *analytic*. *It implies that the propositions of a mathematical theory cannot be deduced from logical laws and*

[66] -----, 2003, "*Kant's Mathematical Epistemology*",Retrieved 2004 <*http://www.wesleyan. edu/phil/ courses/202/s00/pdfs/phil202s0011a.pdf.*>

[67] ….., "*Immanuel Kant, 1724–1804*", Retrieved 2004 <*http://www.alcott.net/alcott/ home/champions/ Kant.html*>

[68] Wilder, R. L. , 1952, *"Introduction to the Foundation of Mathematics",* New York, p. 198

definitions. Space[69] is represented as *a pure intuition* by showing that representation provides us with a way to structure *empirical intuitions.* Shabel L. clarifies Kant's notion of a particular feature of the concept of *space* i.e. the form of outer sense. It is able to account for the features of *geometric cognition* i.e. *the synthetic a priori* of geometric cognition. While, *space,* as form of sensible intuition, is able to account for the *applicability* of geometric cognition. If[70] the *pure intuition of space* that affords cognition of the principles of geometry were not the form of our *outer* then the *principles of geometry* would have no role as a science of *spatial objects.*

According to Kant[71], mathematics depends on those of *space* and *time* that means that the *abstract extension of the mathematical forms* embodied in our experience parallels a*n extension of the objective world* beyond what we actually perceive. Wilder R.L. points out the arguments for the claim that intuition plays an essential role in *mathematics* are inevitably *subjectivist* to a degree, in that they pass from a direct consideration of the *mathematical statements* and of what is required for their truth verifying them. The dependence of *mathematics*[72] on *sensible intuition* gives some plausibility to the view that the possibility of *mathematical representation* rests on the *form of our sensible intuition.* This conception[73] could be extended to the *intuitive verification* of elementary propositions of the arithmetic of *small numbers*. If

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[69] *Ibid.p. 198*

[70] Shabel, L., 1998, "*Kant's "Argument from Geometry*", Journal of the History of Philosophy, The Ohio State University, p.20

[71] Wilder, R. L. , 1952, *"Introduction to the Foundation of Mathematics",* New York, p.198

[72] Ibid. p.198

[73] *Ibid. p.198*

these propositions really are evident in their *full generality,* and hence are *necessary*, then this conception gives some insight into the nature of this evidence.

According to Wilder R.L. [74], Kant connects *arithmetic* with *time* as the *form of our inner intuition*, although he did not intend by this to deny that there is no direct reference to *time* in *arithmetic*. The claim[75] apparently is that to a fully explicit awareness of number goes the *successive apprehension* of the stages in its construction, so that the structure involved is also represented by *a sequence of moments* of *time*. *Time*[76] thus provides a realization for any number which can be realized in experience at all. Although this view[77] is plausible enough, it does not seem strictly necessary to preserve the connection with *time* in the *necessary extrapolation* beyond *actual experience*. Wilder R.L. [78] sums up that thinking of *mathematical construction* as a process in *time* is a useful picture for interpreting problems of *constructivity* the *mathematical concepts.*

While, Palmquist, S.P. in "*Kant On Euclid: Geometry In Perspective"* describes that, as for Kant, *space* is the *pure form of our sensible intuition*. The implication of this theory is that the *intuitive character of mathematics* is limited to objects which can be constructed. In other words[79], Kant's mature position is that *intuition* limits *the broader region* of logical existence to *the narrower region* of

[74] *Ibid*.p.198
[75] *Ibid. p. 198*
[76] *Ibid. p.198*
[77] *Ibid.p.198*
[78] *Ibid. p.198*
[79] Palmquist, S.P., 2004, "*Kant On Euclid: Geometry In Perspective*", Retreived 2004 < *Steve Pq @hkbu.edu.hk>)*

mathematical existence. There[80] can be no doubt that it is clear to Kant that in *geometry*, the field of what is *logically possible* extends far beyond that of *Euclidean geometry*. Palmquist, S.P. (2004) states the following:

> Under the Kant's presuppositions it is not only possible but necessary to assume the existence of non-Euclidean geometries because non-Euclidean geometries are not only logically possible but also they cannot be constructed; hence they have no real mathematical existence for Kant and are mere figments of thought[81]

Palmquist, S.P. sums up that Kant's view enables us to obtain a more accurate picture of the role of *intuition* in *mathematics.* On the other hand, Wilder R.L. alleges that Kant went on to maintain that *the evidence* of both the principles of geometry and those of arithmetic rested on the *form of our sensible intuition.* In particular[82], he says that mathematical demonstrations proceeded by *construction of concepts in pure intuition,* and thus they appealed to the form of *sensible intuition*.

Other writer, Johnstone H.W. in Sellar W. ascribes that Kant's sensible intuition account the role in *foundation of mathematics* by the productive imagination in perceptual geometrical shapes. *Phenomenological* reflection[83] on the structure of perceptual geometrical shapes, therefore, should reveal the *categories,* to which these objects belong, as well as the manner in which objects perceived and perceiving subjects come together in the perceptual act. To dwell it we need to consider Kant's distinction between (a) *the concept of an object,* (b) *the schema of the concept*, and (c) *an image of the object,* as well as his explication of the distinction between *a*

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[80] *Ibid.*
[81] *Ibid.*
[82] Wilder, R. L. , 1952, *"Introduction to the Foundation of Mathematics",* New York, p.198
[83] Johnstone, H.W., 1978, in Sellars, W., 1978 "*The Role Of The Imagination In Kant's Theory Of Experience*", Retrieved 2003 < *http://www.ditext.com/index.html*>

geometrical shape as object and *the successive manifold* in the apprehension of a geometrical shape.

Johnstone H.W. [84] indicates that the *geometrical object* is that the *appearance* which contains the condition of this *necessary rule of apprehension* and the productive imagination which generates the complex demonstrative conceptualization. Bolzano B. (1810) acknowledges that Kant found a great difference between the *intuition* in which some sketched triangle actually produces, and a triangle constructed only in the imagination. Bolzano B. [85] states that the *first* as altogether *superfluous* and *insufficient* for the proof of an *synthetic a priori* proporsition, but the *latter* as *neccessary* and *sufficient.* According to Johnstone H.W.[86]Kant's sensible intuitions in mathematics are complex demonstrative thoughts which have implicit *categorical form.* Kant [87] emphasizes the difference between *intuitions* on the one hand and *sensations* and *images* on the other. *It is intuitions and not sensations or images which contain categorical form.*

Johnstone H.W.[88] highlights Kant's notion that the *synthesis* in connection with perception has two things in mind (1) *the construction of mathematical model as an image,* (2) *the intuitive formation of mathematical representations as a complex demonstratives.* Since *mathematical intuitions* have *categorical form,* we can find this

[84] *Ibid*.
[85] Bolzano, B., 1810, "*Appendix: On the Kantian Theory of the Construction of Concepts through Intuitions"* in Ewald, W., 1996, "*From Kant to Hilbert: A Source Book in the Foundations of Mathematics, Volume I",* Oxford: Clarendon Press, p.219-221
[86] Johnstone, H.W., 1978, in Sellars, W., 1978 "*The Role Of The Imagination In Kant's Theory Of Experience",* Retrieved 2003 < *http://www.ditext.com/index.html*>
[87] *Ibid.*
[88] *Ibid*

categorical form in them and arrive at *categorical concepts of mathematics* by abstracting from experience. Meanwhile, Kant in "*The Critic Of Pure Reason: APPENDIX*" states :

> It would not even be necessary that there should be only one straight line between two points, though experience invariably shows this to be so. What is derived from experience has only comparative universality, namely, that which is obtained through induction.We should therefore only be able to say that, so far as hitherto observed, no space has been found which has more than three dimensions [89]

Shapiro[90] claims that for the *dependence intuition*, ordinary physical objects are *ontologically independent*, not only of us, but of each other. The existence of the *natural number 2*, for instance, appears not to involve that of the *empty set*, nor *vice versa.* According to Shapiro[91], the d*ependence intuition* denies that *mathematical objects* from the same structure are *ontologically independent* of each other in this way. The existence of the *natural number 2,* for instance, depends upon other natural numbers. It makes no sense to say that *2* could have existed even if *5* did not. Shapiro[92] suggests that *the natural number structure* is prior to its *individual elements*, such that if one element *exists*, all do. However, he admits that it is hard to give a satisfactory explication of the d*ependence intuition*, since *pure mathematical objects exist necessarily* and the usual modal explication of ontological dependence gets no

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[89] Kant, I., 1781, "*The Critic Of Pure Reason: APPENDIX*" Translated By J. M. D. Meiklejohn, Retrieved 2003<*http://www.encarta.msn. com/*>

[90] Linnebo, Ø., 2003, "*Review of Stewart Shapiro, Philosophy of Mathematics: Structure and Ontology*", Retrieved 2004 < *http://www.oystein.linnebo@filosofi.uio.no*>

[91] *Ibid.*

[92] In Linnebo, Ø., 2003, "*Review of Stewart Shapiro, Philosophy of Mathematics: Structure and Ontology*", Retrieved 2004 < *http://www.oystein.linnebo@filosofi.uio.no*>

foothold. For on this explication, the existence of *2* no more depends on that of 5 than on that of the empty set. Shapiro stated the following:

> There are two possible sensein which *category* theory could serve as a foundations for mathematics: the strong sense i.e. all mathematical concepts, including those of the current, *logico-meta-theoretical* framework for mathematics, are explicable in *category*-theoretic terms; and the weaker sense i.e. one only requires *category* theory to serve as a possibly superior substitute for axiomatic set theory in its present foundational role. [93]

Bell[94] argues that it is implausible that *category theory* could function as a foundation in the *strong sense*, because even set theory does not serve this function. This is due to the fact that *set theory* is *extensional*, and the combinatorial aspects of mathematics, which is concerned with the finitely presented properties of the inscriptions of the formal language, is *intentional*. Bell[95] claims that this branch deals with objects such as *proofs* and constructions whose actual presentation is crucial.

Further, Shapiro [96] claims that for the *structuralist intuition*, the Scarce Properties Intuition has probably been the primary motivation for the recent wave of interest in *mathematical structuralism.* This *intuition*[97] says *there is no more to the individual numbers "in themselves" than the relations they bear to each other*. The numbers have no '*internal composition*' or *extra-structural properties*; rather, all the properties they have are those they have in virtue of occupying positions in the natural

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[93] *Ibid*.
[94] In Linnebo, Ø., 2003, "*Review of Stewart Shapiro, Philosophy of Mathematics: Structure and Ontology*", Retrieved 2004 < *http://www.oystein.linnebo@filosofi.uio.no*>
[95] *Ibid.*
[96] *Ibid*.
[97] *Ibid*

number structure. A natural explication of the *Scarce Properties Intuition*[98] is that *the natural numbers* have only arithmetical properties and that, for this reason, science should be regimented in a *many-sorted language*, where arithmetical expressions form a sort of their own. Metaphysically[99], this would correspond to the claim that *the natural numbers* form their own *category*. Shapiro seems quite sympathetic with this explication.

One argument[100] is that on Shapiro's version of *structuralism* there is *a plethora of mathematical structures* that says *"not only natural numbers but integers, rationals, reals, complex numbers, quaternions, and so on through the vast zoology of non-algebraic structures that modern mathematics provides".* Each of these *structures* has its own *category*. In contrast, there is no such proliferation of categories in the realm of the concrete. So there must be something special about *pure mathematics* that is responsible for this proliferation. A second[101], complimentary, argument is contained in the third *structuralist intuition.* Shapiro[102] draws upon that *the properties of pure mathematical object* are *purely formal*, unlike the substantive properties possessed by concrete objects. Shapiro[103] called this the *formality intuition.* This *intuition*[104] is captured by Shapiro's claim that the subject matter of pure mathematics are *structures*, where a structure is said to be *'the abstract form of a system'* of objects

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[98] *Ibid.*
[99] *Ibid.*
[100] *Ibid*.
[101] *Ibid.*
[102] *Ibid.*
[103] *In* Linnebo, Ø., 2003, "*Review of Stewart Shapiro, Philosophy of Mathematics: Structure and Ontology*", Retrieved 2004 < *http://www.oystein.linnebo@filosofi.uio.no*>
[104] *Ibid.*

and relations on these objects. A *structure*[105] can thus be instantiated by a variety of systems of more substantive objects and relations; for instance, the *natural number structure* can be instantiated by the sequence of ordinary numerals and by the sequence of strokes: |, ||, |||, etc.

Conversely[106], *a structure* can be arrived at by abstraction from a system of more substantive objects and relations. Shapiro[107] makes a very interesting suggestion about what it means for *a property* to be *formal,* as opposed to *substantive.* Recall Tarski's characterization of a logical notion as one whose extension remains unchanged under every permutation of the domain. Drawing on this idea, Shapiro[108] suggests that *a property* is formal just in case it can be completely defined in a higher-order language. It uses only terminology that denotes objects and relations of the system.

D. The Relevance of Kant's Theory of Knowledge to Contemporary Foundation of Mathematics

The relevance of Kant's theory of knowledge to the contemporary foundation of mathematics can be traced from the notions of contemporary writers. Jørgensen, K.F.(2006) admits that *a philosophy of mathematics* must square with *contemporary*

[105] Linnebo, Ø., 2003, "*Review of Stewart Shapiro, Philosophy of Mathematics: Structure and Ontology*", Retrieved 2004 < *http://www.oystein.linnebo@filosofi.uio.no*>
[106] *Ibid.*
[107] *Ibid.*
[108] *Ibid.*

mathematics as it is carried out by actual mathematicians. This[109] leads him to define a very general notion of constructability of mathematics on the basis of a generalized understanding of *Kant's theory of schema*. Jørgensen, K.F. further states that *Kant's theory of schematism* should be taken seriously in order to understand his *Critique*. It was science which Kant wanted to provide a foundation for. He says that one should take *schematism* to be a very central feature of *Kant's theory of knowledge.*

Meanwhile, Hanna, R. insists that Kant offers an account of human rationality which is essentially oriented towards *judgment*. According to her, Kant also offers an account of the nature of *judgment*, the nature of *logic*, and the nature of the various irreducibly different kinds of *judgments,* that are essentially oriented towards the anthropocentric empirical referential meaningfulness and truth of the proposition. Further, Hanna, R.[110] indicates that the rest of *Kant's theory of judgment* is then thoroughly cognitive and non-reductive. In Kant[111], *propositions* are systematically built up out of directly *referential terms (intuitions)* and *attributive* or *descriptive terms (concepts*), by means of unifying acts of our *innate* spontaneous cognitive faculties. This *unification* is based on *pure logical constraints* and under *a higher-order unity* imposed by our faculty for rational *self-consciousness.* Furthermore[112] all of this is consistently combined by Kant with *non-conceptualism* about intuition, which entails that *judgmental rationality* has a *pre-rational* or *proto-rational cognitive*

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[109] Jørgensen, K.F., 2006, "Philosophy of Mathematics" Retrieved 2006 <http://akira.ruk. dk/ ~frovin /work.htm>
[110] Hanna, R., 2004, "Kant's Theory of Judgment", Stanford Encyclopedia of Philosophy, Retrieved 2004, <http://plato.stanford.edu/cgi-bin/encyclopedia/archinfo.cgi?entry=kant-judgment>
[111] Ibid.
[112] *Ibid.*

grounding in more basic *non-conceptual cognitive capacities* that we share with various non-human animals. In these ways, Hanna, R.[113] concludes that *Kant's theory of knowledge* is the *inherent philosophical interest, contemporary relevance,* and *defensibility remain essentially intact* no matter what one may ultimately think about his controversial *metaphysics of transcendental idealism.*

Meanwhile, Hers R. insists that at the bottom tortoise of *Kant's synthetic a priori* lies *intuition*. In the sense of *contemporary foundation of mathematics*, Hers R.[114] notifies that in providing truth and certainty in mathematics Hilbert implicitly referred Kant. He.[115] pointed out that, like Hilbert, Brouwer was sure that *mathematics* had to be established on a sound and *firm foundation* in which *mathematics* must start from the intuitively given. The name *intuitionism*[116] displays its descent from Kant's intuitionist theory of mathematical knowledge. Brouwer follows Kant in saying that *mathematics* is founded on *intuitive truths*. As it was learned that Kant though *geometry* is based on *space intuition*, and *arithmetic* on *time intuition*, that made both geometry and arithmetic "*synthetic a priori*". About geometry, Frege[117] agrees with Kant that it is *synthetic intuition*. Furthermore, Hers R. indicates that all contemporary standard philosophical viewpoints rely on some notions of intuition; and consideration of intuition as actually experienced leads to a

[113] *Ibid.*
[114] Hersh, R., 1997*, "What is Mathematics, Really?"*, London: Jonathan Cape, p.162
[115] *Ibid.p.162*
[116] *Ibid. p.162*
[117] *In* Hersh, R., 1997*, "What is Mathematics, Really?"*, London: Jonathan Cape, p.162

notion that is difficult and complex but not inexplicable. Therefore, Hers R[118] suggests that *a realistic analysis of mathematical intuition* should be a central goal of the philosophy of mathematics.

In the sense of very contemporary practical and technical mathematics works Polya G in Hers R. states:

> Finished mathematics presented in a finished form appears as purely demonstrative, consisting of proofs only. Yet mathematics in the making resembles any other human knowledge in the making. You have to guess a mathematical theorem before you prove it; you have to have the idea of the proof before you carry through the details. You have to combine observations and follow analogies; you have to try and try again.[119]

The writer of this dissertation perceives that we may examine above notions in the frame work of *Kant's theory of knowledge* to prove that it is relevant to the current practice of mathematics. We found some related key words to Kant's notions e.g. "*presented"*, *"appears", "human knowledge", "observation"* and "*analogies".* We may use *Kant's notions* to examine contemporary practice of mathematics e.g. by reflecting metaphorical power of the *"myth"* of the foundation of contemporary mathematics. Hers R. listed the following myth: *1) there is only one mathematics-indivisible now and forever, 2) the mathematics we know is the only mathematics there can be, 3) mathematics has a rigorous method which yields absolutely certain conclusion, 4) mathematical truth is the same for everyone.*

Meanwhile, Mrozek, J. (2004) in *"The Problems of Understanding Mathematics"* attempts to explain contemporary the structure of the process of

[118] *Ibid. p. 216*
[119] *Ibid. p.216*

understanding mathematical objects such as *notions, definitions, theorems,* or *mathematical theories.* Mrozek, J.[120] distinguishes *three basic planes* on which the process of understanding mathematics takes place: *first,* understanding the meaning of notions and terms existing in mathematical considerations i.e. mathematician must have the knowledge of what the given symbols mean and what the corresponding notions denote; *second,* understanding concerns the structure of the object of understanding wherein it is the sense of the sequences of the applied notions and terms that is important; and *third,* understanding the *'role'* of the object of understanding - *consists in fixing the sense of the object of understanding in the context of a greater entity* - i.e., it is an investigation of the background of the problem. Mrozek, J.[121] sums up that *understanding mathematics,* to be sufficiently comprehensive, should take into account at least three other connected considerations - *historical, methodological and philosophical* - as ignoring them results in *a superficial* and *incomplete* understanding of mathematics.

Furthermore, Mrozek, J. [122] recommends that contemporary practice in mathematics could investigate *properly, un-dogmatically* and *non-arbitrarily* the classical problems of philosophy of mathematics as it was elaborated in *Kant's theory of knowledge*. According to him, it[123] implies that *teaching mathematics* should not consist only in inculcating abstract formulas and conducting formalized considerations;

[120] Mrozek, J.,2004, "The Problems of Understanding Mathematics" University of Gdańsk, Gdańsk, Poland. Retieved 2004 <http://www.bu.edu/wcp/papers/math/MathMroz.htm>
[121] *Ibid.*
[122] *Ibid.*
[123] *Ibid.*

we can not learn mathematics without its thorough understanding. Mrozek, J.[124] sums up that in the process of *teaching mathematics*, we should take into account both the history and philosophy (with methodology) of mathematics i.e. theory of knowledge and epistemological foundation of mathematics, since neglecting them makes the understanding of mathematics superficial and incomplete.

[124] *Ibid.*